$A_4^{(1)}$ 型４階パンルベ微分方程式の簡約化において現れる７連立微分方程式系の一般解

A general solution of the 7-system of nonlinear ordinary differential equations appearing in the reduction of the 4th order Painlevé equation of type $A_4^{(1)}$

By

Setsuji YOSHIDA
(Sagami Women's University)

Abstract

In the precedent papers [2] and [3] the author studied the higher order Painlevé equations of type $A_4^{(1)}$, and obtained its formal general solution under the condition $b = 0$. In this article we study a 7-system (E)+(F) which is derived from the reduction of $A_4^{(1)}$, and gives a general solution for (E)+(F) without the restriction $b = 0$. First we solve the 3-system (F) and then we give an explicit solution of the 4-system (E), which leads also to a formal general solution of $A_4^{(1)}$.

Key Words: higher order Painlevé equation, reduction, general solution, $A_4^{(1)}$

§ 1. A general solution of (F)

We study the 7-system of nonlinear ordinary differential equations which consists of 4-system (E) and 3-system (F). It is described by (E)

(1.1) $$x^3 \frac{dU_1}{dx} = U_1(\lambda_1(x) + \alpha x^2 V_1) - \frac{\alpha b}{2} x^2 U_3$$

(1.2) $$x^3 \frac{dU_2}{dx} = U_2(\lambda_2(x) - \alpha x^2 V_1 + \alpha x^2 V_2 - 2\alpha x^2 V_3)$$

(1.3) $$x^3 \frac{dU_3}{dx} = U_3(\lambda_3(x) + 2\alpha x^2 V_1 + \alpha x^2 V_3)$$

(1.4) $$x^3 \frac{dU_4}{dx} = U_4(\lambda_4(x) - \alpha x^2 V_3) + \frac{\alpha b}{2} x^2 U_2 - \alpha x^2 V_1 \cdot U_2$$

and by (F)

(1.5) $$\frac{x}{\alpha}\frac{dV_1}{dx} = V_1(V_2 - 2V_3) - \frac{b}{2}V_2$$

(1.6) $$\frac{x}{\alpha}\frac{dV_2}{dx} = V_2(\beta + V_1 + V_2 - V_3)$$

(1.7) $$\frac{x}{\alpha}\frac{dV_3}{dx} = -V_1(V_2 - 2V_3) + \frac{b}{2}V_2$$

where

(1.8) $$\lambda_j(x) = (-1)^{j-1}\lambda(x) + \alpha_j x^2, \qquad \lambda(x) = -k - fx$$

(1.9) $$\alpha = \frac{4}{k} \ ,\beta = \frac{1}{2}(b + c - d)$$

(1.10) $$\alpha_1 = \frac{\alpha}{2}(a-b)+1, \quad \alpha_2 = -\frac{\alpha}{2}(a-b)+1,$$

$$\alpha_3 = \frac{\alpha}{2}(a+c-d)+1, \quad \alpha_4 = -\frac{\alpha}{2}(a+c-d)+1$$

(1.11) a, b, c, d, k, f , being complex constants.

In a generic case we can assume

(1.12) $k \neq 0$.

A general solution of System (F) First we shall solve the 3-system (F) given by (1.5), (1.6), (1.7).

Noticing $\frac{x}{\alpha}\frac{d}{dx}(V_1 + V_3) = 0$, we see

(1.13) $V_1 + V_3 = C_1$, C_1 being an integration constant.

Then (1.5) and (1.6) are written by

(1.14) $$\frac{x}{\alpha}\frac{dV_1}{dx} = V_1(-2C_1 + 2V_1 + V_2) - \frac{b}{2}V_2$$

(1.15) $$\frac{x}{\alpha}\frac{dV_2}{dx} = V_2(\beta - C_1 + 2V_1 + V_2)$$

We see

$$\frac{x}{\alpha}\frac{d}{dx}(\log V_1 - \log V_2) = -C_1 - \beta - \frac{b}{2}\frac{V_2}{V_1}$$

(1.16) $$\frac{x}{\alpha}\frac{d}{dx}(\log\frac{V_2}{V_1}) = \beta + C_1 + \frac{b}{2}\frac{V_2}{V_1}$$

Put $\dfrac{V_2}{V_1}$ by W, then W satisfies

(1.17) $$\frac{dW}{W(\beta + C_1 + \frac{b}{2}W)} = \frac{\alpha dx}{x}$$

Thus we have

(1.18) $$W \equiv \frac{V_2}{V_1} = (\beta + C_1)\left\{C_2 x^{-\alpha(\beta+C_1)} - \frac{b}{2}\right\}^{-1}$$, C_2 being an integration constant,

and

(1.19) $$V_1 = W^{-1}V_2 = \frac{\left\{C_2 x^{-\alpha(\beta+C_1)} - \frac{b}{2}\right\}}{\beta + C_1} V_2 .$$

By substituting (1.19) into (1.15), we have

(1.20) $$\frac{x}{\alpha}\frac{dV_2}{V_2 dx} = \beta - C_1 + (2W^{-1} + 1)V_2 ,$$

and hence

$$\frac{x}{\alpha}\frac{d}{dx}\left\{\log\left(x^{-\alpha(\beta-C_1)}V_2\right)\right\} = (2W^{-1}+1)V_2$$
$$= (2W^{-1}+1)x^{\alpha(\beta-C_1)} \cdot x^{-\alpha(\beta-C_1)}V_2 .$$

By introducing

(1.21) $$H = x^{-\alpha(\beta-C_1)}V_2 ,$$

we have

(1.22) $$\frac{x}{\alpha}\frac{1}{H^2}\frac{dH}{dx} = (2W^{-1}+1)x^{\alpha(\beta-C_1)}$$

(1.23) $$\frac{dH}{H^2} = \frac{1}{\beta + C_1}\frac{\alpha}{x}\left\{2C_2 x^{-2\alpha C_1} + \left(\beta + C_1 - b\right)x^{\alpha(\beta-C_1)}\right\}dx.$$

Hence

(1.24) $$-H^{-1} = \frac{1}{\beta + C_1}\left(-\frac{C_2}{C_1}x^{-2\alpha C_1} + \frac{\beta + C_1 - b}{\beta - C_1}x^{\alpha(\beta-C_1)} + C_3\right)$$

(1.25) $$H = \left(\beta + C_1\right)\left(\frac{C_2}{C_1}x^{-2\alpha C_1} - \frac{\beta + C_1 - b}{\beta - C_1}x^{\alpha(\beta-C_1)} - C_3\right)^{-1},$$

C_3 being an integration constant,

and

(1.26) $$V_2 = x^{\alpha(\beta - C_1)} H = \frac{(\beta + C_1)x^{\alpha(\beta - C_1)}}{\frac{C_2}{C_1}x^{-2\alpha C_1} - \frac{\beta + C_1 - b}{\beta - C_1}x^{\alpha(\beta - C_1)} - C_3}$$
$$= \frac{\frac{C_1}{C_2}(\beta + C_1)x^{\alpha(\beta + C_1)}}{1 - \frac{\beta + C_1 - b}{\beta - C_1}\frac{C_1}{C_2}x^{\alpha(\beta + C_1)} - \frac{C_1 C_3}{C_2}x^{2\alpha C_1}}$$

Define $(W_1(x), W_2(x), W_3(x))$ and put integration constants (C_1', C_2', C_3') by

(1.27) $$W_1(x) = C_1 \equiv C_1'$$

(1.28) $$W_2(x) = \frac{(\beta - b + C_1)C_1}{(\beta - C_1)C_2}x^{\alpha(\beta + C_1)} \equiv C_2' x^{\alpha(\beta + C_1)} = C_2' x^{\alpha(\beta + W_1)}$$

(1.29) $$W_3(x) = \frac{C_1 C_3}{C_2}x^{2\alpha C_1} \equiv C_3' x^{2\alpha W_1}.$$

By using (2.4) we obtain the proposition:

Proposition 1 The general solution of the system (F) is given by

(1.30) $$V_1 = \left\{W_1(x) - \frac{b(\beta - W_1(x))}{2(\beta - b + W_1(x))}W_2(x)\right\}(1 - W_2(x) - W_3(x))^{-1}$$

(1.31) $$V_2 = \frac{\beta^2 - W_1(x)^2}{\beta - b + W_1(x)}W_2(x)(1 - W_2(x) - W_3(x))^{-1}$$

(1.32) $$V_3 = W_1(x) - V_1(x)$$
$$= \left\{-W_1(x)W_2(x) + \frac{b(\beta - W_1(x))}{2(\beta - b + W_1(x))}W_2(x) - W_1(x)W_3(x)\right\}(1 - W_2(x) - W_3(x))^{-1}.$$

Remark 1 $(W_1, W_2, W_3) = (W_1(x), W_2(x), W_3(x))$ satisfies the 3-sytem of equations

(1.33) $$\frac{dW_1}{dx}=0$$

(1.34) $$\frac{dW_2}{W_2dx}=\frac{\alpha}{x}(\beta+W_1)$$

(1.35) $$\frac{dW_3}{W_3dx}=\frac{\alpha}{x}2W_1.$$

§ 2. $Q_{\pm}(W_1,W_2,W_3)$

Before we solve the system (E), we define a function $Q_+(W_1,W_2,W_3)$ of (W_1,W_2,W_3) by the power series

(2.1) $$Q_+(W_1,W_2,W_3)=\frac{1}{2}\frac{\beta^2-W_1{}^2}{\beta-b+W_1}\sum_{n=0}^{\infty}\sum_{r=0}^{n}{}_nC_r\frac{W_2{}^{r+1}W_3{}^{n-r}}{(2n-r+1)W_1+(r+1)\beta}.$$

$\beta\mathrm{Q}_+$ and $\beta\mathrm{Q}_-$ denote the sets

(2.2) $$\beta\mathrm{Q}_+=\{z\in C \mid \quad z=\beta q,\quad q \quad \textit{is a positive rational number}\}$$

(2.3) $$\beta\mathrm{Q}_-=\{z\in C \mid \quad z=\beta q,\quad q \quad \textit{is a negative rational number}\},$$

respectively. Then we can verify

Lemma 1$_+$ The power series (2.1) converges uniformly in the domain

(2.4) $$(W_1,W_2,W_3)\in D_+\equiv\{(W_1,W_2,W_3)\in C^3 \mid \quad W_1\notin\beta\mathrm{Q}_-,\quad |W_2|<1,\quad |W_3|<1,\quad |W_2+W_3|<1\}$$

Proof Put

(2.5) $$\sum_{n=0}^{\infty}\sum_{r=0}^{n}{}_nC_r\frac{W_2{}^{r}W_3{}^{n-r}}{(2n-r+1)W_1+(r+1)\beta}=\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}A_{j,k}^{+}(W_1)W_2{}^jW_3{}^k,$$

namely

(2.6) $$A_{j,k}^{+}(W_1)={}_{j+k}C_j\frac{1}{(j+2k+1)W_1+(j+1)\beta}$$

then we have

(2.7) $$\left|\frac{A^{+}_{j+1,k}(W_1)W_2^{\,j+1}W_3^{\,k}}{A^{+}_{j,k}(W_1)W_2^{\,j}W_3^{\,k}}\right| = \left|\frac{{}_{j+k+1}C_{j+1}}{{}_{j+k}C_j}\frac{(j+2k+1)W_1+(j+1)\beta}{(j+2k+2)W_1+(j+2)\beta}W_2\right|$$
$$= \left|\frac{j+k+1}{j+1}\cdot\frac{(j+2k+1)W_1+(j+1)\beta}{(j+2k+2)W_1+(j+2)\beta}\right|\cdot|W_2|$$

Therefore if (2.4) is satisfied, we see the (2.7) tends to $|W_2|<1$ as $j\to\infty$, and similarly

(2.8) $$\left|\frac{A^{+}_{j,k+1}\ (W_1)W_2^{\,j}W_3^{\,k+1}}{A^{+}_{j,k}\ (W_1)W_2^{\,j}W_3^{\,k}}\right| = \left|\frac{{}_{j+k+1}C_j}{{}_{j+k}C_j}\frac{(j+2k+1)W_1+(j+1)\beta}{(j+2k+2)W_1+(j+1)\beta}W_3\right|$$
$$= \left|\frac{j+k+1}{k+1}\cdot\frac{(j+2k+1)W_1+(j+1)\beta}{(j+2k+2)W_1+(j+1)\beta}\right|\cdot|W_3|$$

tends to $|W_3|<1$ as $k\to\infty$

Thus we obtain the lemma.

It follows from this lemma, we can show

Proposition 2₊ For the solution $(W_1,W_2,W_3)=(W_1(x),W_2(x),W_3(x))\in D_+$ of (1.33),(1.34),(1.35)

$Q_+(W_1(\mathrm{x}),W_2(\mathrm{x}),W_3(x))$ satisfies

(2.9) $$\frac{dQ_+(W_1(\mathrm{x}),W_2(\mathrm{x}),W_3(x))}{dx} = \frac{\alpha}{2x}V_2(W_1(\mathrm{x}),W_2(\mathrm{x}),W_3(x))$$

Proof

$$\frac{2x}{\alpha}\frac{dQ_+(W_1(\mathrm{x}),W_2(\mathrm{x}),W_3(x))}{dx} = \frac{2x}{\alpha}\left(\frac{\partial Q_+}{\partial W_1}\frac{dW_1}{dx}+\frac{\partial Q_+}{\partial W_2}\frac{dW_2}{dx}+\frac{\partial Q_+}{\partial W_3}\frac{dW_3}{dx}\right)$$
$$= \frac{\beta^2-W_1^{\,2}}{\beta-b+W_1}\sum_{n=0}^{\infty}\sum_{r=0}^{n}{}_nC_r\frac{(r+1)(\beta+W_1)+2(n-r)W_1}{(2n-r+1)W_1+(r+1)\beta}W_2^{\,r+1}W_3^{\,n-r}$$
$$= \frac{\beta^2-W_1^{\,2}}{\beta-b+W_1}W_2\sum_{n=0}^{\infty}\sum_{r=0}^{n}{}_nC_rW_2^{\,r}W_3^{\,n-r} = \frac{\beta^2-W_1^{\,2}}{\beta-b+W_1}W_2\sum_{n=0}^{\infty}(W_2+W_3)^n$$
$$= \frac{\beta^2-W_1^{\,2}}{\beta-b+W_1}\cdot\frac{W_2}{1-W_2-W_3} = V_2(W_1(\mathrm{x}),W_2(\mathrm{x}),W_3(x))$$

Remark 2₊ Assume that $\mathrm{Re}(xW_1) > 0$ and $\mathrm{Re}(\alpha\beta) > 0$. When x tends to 0 in a sector with vertex $x = 0$, then $W_2(\mathrm{x})$ and $W_3(\mathrm{x})$ tends to 0, if $|W_1| = |C_1|$ is sufficiently small. Moreover we see that $Q_+(W_1(\mathrm{x}), W_2(\mathrm{x}), W_3(x))$ tends to 0 by using $Q_+(W_1, W_2, W_3) = O(W_2)$ near $(W_1, W_2, W_3) = (0,0,0)$.

Next we define a function $Q_-(W_1, W_2, W_3)$ of (W_1, W_2, W_3) by the power series

$$(2.10)\quad Q_-(W_1, W_2, W_3) = \frac{1}{2}\frac{\beta^2 - W_1^{\,2}}{\beta - b + W_1}\sum_{n=0}^{\infty}\sum_{r=0}^{n} {}_nC_r \frac{\left(-\frac{W_2}{W_3}\right)^{r+1}\left(\frac{1}{W_3}\right)^{n-r}}{(-2n + r - 1)W_1 + (r+1)\beta}.$$

As in Lemma 1₊ we can verify

Lemma 1₋ The power series (2.10) converges uniformly in the domain

$$(2.11)\quad (W_1, W_2, W_3) \in D_- \equiv \left\{(W_1, W_2, W_3) \in C^3 \middle| \ W_1 \notin \beta\mathrm{Q}_+ \ ,\ \left|\frac{W_2}{W_3}\right| < 1,\ \left|\frac{1}{W_3}\right| < 1,\ \left|\frac{W_2}{W_3} - \frac{1}{W_3}\right| < 1\right\}$$

Proof Put

$$(2.12)\quad \sum_{n=0}^{\infty}\sum_{r=0}^{n} {}_nC_r \frac{\left(-\frac{W_2}{W_3}\right)^{r}\left(\frac{1}{W_3}\right)^{n-r}}{(-2n + r - 1)W_1 + (r+1)\beta} = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} A^-_{j,k}(W_1)\left(-\frac{W_2}{W_3}\right)^{j}\left(\frac{1}{W_3}\right)^{k},$$

namely

$$(2.13)\quad A^-_{j,k}(W_1) = {}_{j+k}C_j \frac{1}{(j + 2k + 1)W_1 + (j+1)\beta}$$

then we have

$$(2.14)\quad \left|\frac{A^-_{j+1,k}(W_1)\left(-\frac{W_2}{W_3}\right)^{j+1}\left(\frac{1}{W_3}\right)^{k}}{A^-_{j,k}(W_1)\left(-\frac{W_2}{W_3}\right)^{j}\left(\frac{1}{W_3}\right)^{k}}\right| = \left|\frac{{}_{j+k+1}C_{j+1}}{{}_{j+k}C_j}\cdot\frac{(j + 2k + 1)W_1 + (j+1)\beta}{(j + 2k + 2)W_1 + (j+2)\beta}\cdot\frac{W_2}{W_3}\right|$$

$$= \left|\frac{j + k + 1}{j+1}\cdot\frac{(j + 2k + 1)W_1 + (j+1)\beta}{(j + 2k + 2)W_1 + (j+2)\beta}\right|\cdot\left|\frac{W_2}{W_3}\right|$$

Therefore if (2.11) is satisfied, we see the (2.10) tends to $\left|\frac{W_2}{W_3}\right|<1$ as $j\to\infty$, and similarly

$$(2.15)\quad \left|\frac{A_{j,k+1}(W_1)\left(-\frac{W_2}{W_3}\right)^j\left(\frac{1}{W_3}\right)^{k+1}}{A_{j,k}(W_1)\left(-\frac{W_2}{W_3}\right)^j\left(\frac{1}{W_3}\right)^{k}}\right|=\left|\frac{{}_{j+k+1}C_j}{{}_{j+k}C_j}\cdot\frac{(j+2k+1)W_1+(j+1)\beta}{(j+2k+2)W_1+(j+1)\beta}\cdot\frac{1}{W_3}\right|$$

$$=\left|\frac{j+k+1}{k+1}\cdot\frac{(j+2k+1)W_1+(j+1)\beta}{(j+2k+2)W_1+(j+1)\beta}\right|\cdot\left|\frac{1}{W_3}\right|$$

tends to $\left|\frac{1}{W_3}\right|<1$ as $k\to\infty$.

Proposition 2₋ For the solution $(W_1,W_2,W_3)=(W_1(x),W_2(x),W_3(x))\in D_-$ of (1.33),(1.34),(1.35),

$Q_-(W_1(\mathrm{x}),W_2(\mathrm{x}),W_3(x))$ satisfies

$$(2.16)\quad \frac{dQ_-(W_1(\mathrm{x}),W_2(\mathrm{x}),W_3(x))}{dx}=\frac{\alpha}{2x}V_2(W_1(\mathrm{x}),W_2(\mathrm{x}),W_3(x))$$

Proof Since $Q_-(W_1,W_2,W_3)$ is written by

$$(2.17)\quad Q_-(W_1,W_2,W_3)=\frac{1}{2}\frac{\beta^2-W_1^{\ 2}}{\beta-b+W_1}\sum_{n=0}^{\infty}\sum_{r=0}^{n}{}_nC_r\frac{(-W_2)^{r+1}W_3^{\ -(n+1)}}{(-2n+r-1)W_1+(r+1)\beta},$$

we have

$$\frac{2x}{\alpha}\frac{dQ_-(W_1(\mathrm{x}),W_2(\mathrm{x}),W_3(x))}{dx}=\frac{2x}{\alpha}\left(\frac{\partial Q_+}{\partial W_1}\frac{dW_1}{dx}+\frac{\partial Q_+}{\partial W_2}\frac{dW_2}{dx}+\frac{\partial Q_+}{\partial W_3}\frac{dW_3}{dx}\right)$$

$$=\frac{\beta^2-W_1^{\ 2}}{\beta-b+W_1}\sum_{n=0}^{\infty}\sum_{r=0}^{n}{}_nC_r\frac{(r+1)(\beta+W_1)-2(n+1)W_1}{(-2n+r-1)W_1+(r+1)\beta}(-W_2)^{r+1}W_3^{\ -(n+1)}$$

$$=\frac{\beta^2-W_1^{\ 2}}{\beta-b+W_1}\sum_{n=0}^{\infty}\sum_{r=0}^{n}{}_nC_r\left(-\frac{W_2}{W_3}\right)^{r+1}\left(\frac{1}{W_3}\right)^{n-r}$$

$$=\frac{\beta^2-W_1^{\ 2}}{\beta-b+W_1}\left(-\frac{W_2}{W_3}\right)\sum_{n=0}^{\infty}\left(-\frac{W_2}{W_3}+\frac{1}{W_3}\right)^n$$

$$=\frac{\beta^2-W_1^{\ 2}}{\beta-b+W_1}\cdot\frac{-W_2}{W_3}\cdot\frac{1}{1+\frac{W_2}{W_3}-\frac{1}{W_3}}=\frac{\beta^2-W_1^{\ 2}}{\beta-b+W_1}\cdot\frac{-W_2}{W_3+W_2-1}$$

$$=V_2(W_1(\mathrm{x}),W_2(\mathrm{x}),W_3(x))$$

Remark 2_ Assume that $\mathrm{Re}(xW_1)<0$ and $\mathrm{Re}(\alpha\beta)>0$. When x tends to 0 in a sector with vertex $x=0$, then $\frac{W_2(x)}{W_3(x)}$ and $\frac{1}{W_3(x)}$ tends to 0, if $|W_1|=|C_1|$ is sufficiently small. Moreover we see that $Q_-(W_1(\mathrm{x}),W_2(\mathrm{x}),W_3(x))$ tends to 0 by using $Q_-(W_1,W_2,W_3)=O(\frac{W_2}{W_3})$ near $(W_1,\frac{W_2}{W_3},\frac{1}{W_3})=(0,0,0)$.

As for V_1 and V_2 we should notice

Proposition 3

(2.18) $$V_1(W_1,W_2,W_3)=W_1+\frac{1}{2}\frac{(\beta+W_1)W_2+2W_1W_3}{1-W_2-W_3}-\frac{1}{2}V_2(W_1,W_2,W_3)$$

Proof By (1.30) we have

(2.19) $$\begin{aligned}(1-W_2-W_3)V_1 &= W_1-\frac{b(\beta-W_1)}{2(\beta-b+W_1)}W_2\\ &= W_1(1-W_2-W_3)+W_1W_2+W_1W_3-\frac{b(\beta-W_1)}{2(\beta-b+W_1)}W_2\\ &= W_1(1-W_2-W_3)+\frac{(\beta+W_1)W_2+2W_1W_3}{2}-\frac{W_2}{2}\left\{(\beta-W_1)+\frac{b(\beta-W_1)}{\beta-b+W_1}\right\}\\ &= W_1(1-W_2-W_3)+\frac{(\beta+W_1)W_2+2W_1W_3}{2}-\frac{W_2(\beta^2-W_1{}^2)}{2(\beta-b+W_1)}\end{aligned}$$

Remark 3 If (W_1,W_2,W_3) satisfies (1.33), (1.34),(1.35), it is easy to verify

(2.20) $$\frac{d}{dx}\log(1-W_2-W_3)=-\frac{\alpha}{x}\frac{(\beta+W_1)W_2+2W_1W_3}{1-W_2-W_3}$$

§ 3. A general solution of (E)

We define a function $\Lambda(x)$ by

(3.1) $$\Lambda(x)=\int_\infty^x\frac{\lambda(x)}{x^3}dx=\frac{k}{2x^2}+\frac{f}{x}.$$

U_2: **Solution of (1.2)** First we shall solve the equation (1.2) .

From Proposition 1 and Proposition 3 it follows that

$$(3.2)\quad -V_1+V_2-2V_3=-V_1+V_2-2(W_1-V_1)=-2W_1+V_1+V_2$$
$$=-2W_1+\left(W_1+\frac{1}{2}\frac{(\beta+W_1)W_2+2W_1W_3}{1-W_2-W_3}-\frac{1}{2}V_2\right)+V_2$$
$$=-W_1+\frac{1}{2}\frac{(\beta+W_1)W_2+2W_1W_3}{1-W_2-W_3}+\frac{1}{2}V_2 \quad .$$

Hence we have

$$(3.3)\quad \frac{dU_2}{U_2dx}=\frac{\lambda_2(x)}{x^3}+\frac{\alpha}{x}\left(-W_1+\frac{1}{2}\frac{(\beta+W_1)W_2+2W_1W_3}{1-W_2-W_3}+\frac{1}{2}V_2\right)$$

$$\log U_2=-\Lambda(x)+\left(-\frac{\alpha}{2}(a-b)+1\right)\log x-\frac{1}{2}\log\{W_3(1-W_2-W_3)\}+Q_\pm(W_1,W_2,W_3)+const.$$

$$(3.4)\quad U_2=K_2\exp(-\Lambda(x))x^{-\frac{\alpha}{2}(a-b)+1}\left(\frac{1}{W_3(x)(1-W_2(x)-W_3(x))}\right)^{\frac{1}{2}}\exp(Q_\pm(W_1(x),W_2(x),W_3(x))),$$

K_2 being an integration constant.

U_3: **Solution of (1.3)** Next we solve the equation (1.3) . The similar calculations above yields

$$(3.5)\quad 2V_1+V_3=2W_1+\frac{1}{2}\frac{(\beta+W_1)W_2+2W_1W_3}{1-W_2-W_3}-\frac{1}{2}V_2$$

By making use of

$$(3.6)\quad \frac{\alpha}{2}(a+c-d)=\frac{\alpha}{2}(a-b)+\alpha\beta\,,$$

we obtain

$$(3.7)\quad U_3=K_3\exp(\Lambda(x))x^{\frac{\alpha}{2}(a-b)+1}W_2\left(\frac{W_3(x)}{1-W_2(x)-W_3(x)}\right)^{\frac{1}{2}}\exp(-Q_\pm(W_1(x),W_2(x),W_3(x))),$$

K_3 being an integration constant.

U_1: **Solution of (1.1)** In order to solve (1.1), put

(3.8) $$\hat{U}_1 = \exp(\Lambda(x)) x^{\frac{\alpha}{2}(a-b)+1} \left(\frac{W_3(x)}{1 - W_2(x) - W_3(x)} \right)^{\frac{1}{2}} \exp(-Q_{\pm}(W_1, W_2, W_3)),$$

and define

(3.9) $$U_1^* = U_1 \hat{U}_1^{-1}$$

then $\hat{U}_1$ satisfies

(3.10) $$\frac{d\hat{U}_1}{\hat{U}_1 dx} = \frac{1}{x^3}(\lambda_1(x) + \alpha x^2 V_1(x))$$

and

(3.11) $$\frac{dU_1^*}{U_1^* dx} = \frac{dU_1}{U_1 dx} - \frac{d\hat{U}_1}{\hat{U}_1 dx} = -\frac{\alpha b}{2x} \cdot \frac{U_3}{U_1} = -\frac{\alpha b K_3}{2x} \cdot \frac{W_2}{U_1^*}$$

Hence

(3.12) $$U_1^* = -\frac{bK_3}{2} \int \frac{\alpha W_2}{x} dx = -\frac{bK_3}{2} \cdot \frac{W_2}{\beta + W_1} + const.$$

Thus we obtain

(3.13) $$U_1 = K_1 \exp(\Lambda(x)) x^{\frac{\alpha}{2}(a-b)+1} \left(\frac{W_3(x)}{1 - W_2(x) - W_3(x)} \right)^{\frac{1}{2}} \exp(-Q_{\pm}(W_1(x), W_2(x), W_3(x)))$$
$$- \frac{bK_3}{2} \cdot \frac{W_2}{\beta + W_1} \cdot \exp(\Lambda(x)) x^{\frac{\alpha}{2}(a-b)+1} \left(\frac{W_3(x)}{1 - W_2(x) - W_3(x)} \right)^{\frac{1}{2}} \exp(-Q_{\pm}(W_1(x), W_2(x), W_3(x))),$$

K_1 being an integration constant.

U_4: **Solution of (1.4)** In order to solve (1.4) put

(3.14) $$U_4 = \frac{x^2 V_3}{U_3} F(x).$$

Then we have

(3.15) $$\log F(x) = \log U_3 + \log U_4 - 2\log x - \log V_3$$

$$x^3 \frac{d}{dx} \log F(x) = (\lambda_3(x) + 2\alpha x^2 V_1 + \alpha x^2 V_3) + (\lambda_4(x) - \alpha x^2 V_3) + \alpha x^2 \frac{U_2}{U_4} \left(\frac{b}{2} - V_1 \right) - 2x^2$$

$$-\frac{\alpha x^2}{V_3}(-V_1V_2+2V_1V_3+\frac{b}{2}V_2)$$

$$=2\alpha V_1+\alpha x^2\frac{U_2}{U_4}(\frac{b}{2}-V_1)+\alpha x^2(\frac{V_1V_2}{V_3}-2V_1-\frac{b}{2}\frac{V_2}{V_3})$$

$$=\alpha x^2(\frac{b}{2}-V_1)\left(\frac{U_2}{U_4}-\frac{V_2}{V_3}\right)$$

$$\therefore \quad \frac{x}{\alpha}\frac{dF(x)}{F(x)dx}=(\frac{b}{2}-V_1)\left(\frac{U_2U_3}{x^2V_3F(x)}-\frac{V_2}{V_3}\right)=(\frac{b}{2}-V_1)\frac{V_2}{V_3}(\frac{U_2U_3}{x^2V_2F(x)}-1)$$

$$(3.16) \qquad \frac{x}{\alpha}\frac{dF(x)}{dx}=(\frac{b}{2}-V_1)\frac{V_2}{V_3}(\frac{U_2U_3}{x^2V_2}-F(x))$$

By (1.31), (3.4), (3.7) we have

$$(3.17) \qquad \frac{U_2U_3}{x^2V_3}\equiv K_2K_3\cdot\frac{\beta-b+W_1}{\beta^2-W_1{}^2}.$$

Thus (3.16) has a particular solution

$$(3.18) \qquad F\equiv C_0\equiv K_2K_3\cdot\frac{\beta-b+W_1}{\beta^2-W_1{}^2},$$

Put

$$(3.19) \qquad F=G+C_0.$$

Then we have

$$(3.20) \qquad \frac{dG}{Gdx}=\frac{\alpha}{x}(V_1-\frac{b}{2})\frac{V_2}{V_3}$$

$$(3.21) \qquad \log G=\int\frac{\alpha}{x}(V_1-\frac{b}{2})\frac{V_2}{V_3}dx$$

We see by Proposition 1 that

$$(V_1-\frac{b}{2})\frac{V_2}{V_3}=(W_1-V_3-\frac{b}{2})\frac{V_2}{V_3}=-V_2+(W_1-\frac{b}{2})\cdot\frac{V_2}{V_3}$$

Further

$$\frac{V_2}{V_3}=\frac{(\beta^2-W_1{}^2)W_2}{\beta-b+W_1}\cdot\frac{2(\beta-b+W_1)}{-2(\beta-b+W_1)W_1(W_2+W_3)+b(\beta-W_1)W_2}$$

$$=\frac{2(\beta^2-W_1{}^2)W_2}{-2(\beta-b+W_1)W_1(W_2+W_3)+b(\beta-W_1)W_2}$$

$$=\frac{2(\beta^2-W_1{}^2)\dfrac{W_2}{W_3}}{-2(\beta-b+W_1)W_1(\dfrac{W_2}{W_3}+1)+b(\beta-W_1)\dfrac{W_2}{W_3}}$$

$$=\frac{2(\beta^2-W_1{}^2)\dfrac{W_2}{W_3}}{-2(\beta-b+W_1)W_1+(\beta+W_1)(b-2W_1)\dfrac{W_2}{W_3}}$$

By (1.34) and (1.35) $\dfrac{W_2}{W_3}$ satisfies

(3.22) $$\frac{d}{dx}\log\frac{W_2}{W_3}=(\beta-W_1)\frac{\alpha}{x}$$

Hence

$$\frac{\alpha}{x}\frac{V_2}{V_3}dx=\frac{\alpha}{x}\cdot\frac{2(\beta^2-W_1{}^2)\dfrac{W_2}{W_3}}{-2(\beta-b+W_1)W_1+(\beta+W_1)(b-2W_1)\dfrac{W_2}{W_3}}dx$$

$$=\frac{2(\beta^2-W_1{}^2)}{-2(\beta-b+W_1)W_1+(\beta+W_1)(b-2W_1)\dfrac{W_2}{W_3}}\cdot\frac{1}{\beta-W_1}d\left(\frac{W_2}{W_3}\right)$$

(3.23) $$\frac{\alpha}{x}\frac{V_2}{V_3}dx=\frac{2(\beta+W_1)}{-2(\beta-b+W_1)W_1+(\beta+W_1)(b-2W_1)\dfrac{W_2}{W_3}}d\left(\frac{W_2}{W_3}\right)$$

Thus we have

$$\log G=-2Q_\pm+\frac{(W_1-\dfrac{b}{2})2(\beta+W_1)}{(\beta+W_1)(b-2W_1)}\log\left\{-2(\beta-b+W_1)W_1+(\beta+W_1)(b-2W_1)\frac{W_2}{W_3}\right\}+const$$

$$=-2Q_\pm-\log\left\{-2(\beta-b+W_1)W_1+(\beta+W_1)(b-2W_1)\frac{W_2}{W_3}\right\}+const$$

(3.24) $$G=const\cdot\exp(-2Q_\pm)\frac{W_3}{2(\beta-b+W_1)W_1W_3+(\beta+W_1)(2W_1-b)W_2}$$

and

$$(3.25)\qquad U_4=\frac{x^2V_3}{U_3}F(x)$$

$$=\frac{-W_1W_2+\dfrac{b(\beta-W_1)}{2(\beta-b+W_1)}W_2-W_1W_3}{1-W_2-W_3}\cdot K_3^{-1}\exp(-\Lambda(x))x^{-\frac{\alpha}{2}(a-b)+1}W_2^{-1}\left(\frac{W_3}{1-W_2-W_3}\right)^{-\frac{1}{2}}\exp(Q_\pm)$$

$$\cdot\left\{const\cdot\exp(-2Q_\pm)\frac{W_3}{2(\beta-b+W_1)W_1W_3+(\beta+W_1)(2W_1-b)W_2}+C_0\right\}$$

$$=\frac{K_2(\beta-b+W_1)}{\beta^2-W_1^{\,2}}\exp(-\Lambda(x)+Q_\pm)x^{-\frac{\alpha}{2}(a-b)+1}\cdot\frac{-2(\beta-b+W_1)W_1(W_2+W_3)+b(\beta-W_1)W_2}{2(\beta-b+W_1)W_2W_3^{\frac{1}{2}}(1-W_2-W_3)^{\frac{1}{2}}}$$

$$+K_4\exp(-\Lambda(x)-Q_\pm)x^{-\frac{\alpha}{2}(a-b)+1}\frac{-2(\beta-b+W_1)W_1(W_2+W_3)+b(\beta-W_1)W_2}{2(\beta-b+W_1)W_1W_3+(\beta+W_1)(2W_1-b)W_2}$$

$$\cdot\frac{W_3^{\frac{1}{2}}}{2(\beta-b+W_1)W_2(1-W_2-W_3)^{\frac{1}{2}}}\quad ,$$

K_4 being an integration constant.

Thus we have obtained the main theorem in this paper:

Theorem A general solution of (E)+(F) is described by

$$(3.13)\qquad U_1=K_1\exp(\Lambda(x))x^{\frac{\alpha}{2}(a-b)+1}\left(\frac{W_3}{1-W_2-W_3}\right)^{\frac{1}{2}}\exp(-Q_\pm(W_1,W_2,W_3))$$

$$-\frac{bK_3}{2}\cdot\frac{W_2}{\beta+W_1}\cdot\exp(\Lambda(x))x^{\frac{\alpha}{2}(a-b)+1}\left(\frac{W_3}{1-W_2-W_3}\right)^{\frac{1}{2}}\exp(-Q_\pm(W_1,W_2,W_3))$$

$$(3.4)\qquad U_2=K_2\exp(-\Lambda(x))x^{-\frac{\alpha}{2}(a-b)+1}\left(\frac{1}{W_3(1-W_2-W_3)}\right)^{\frac{1}{2}}\exp(Q_\pm(W_1,W_2,W_3))$$

$$(3.7)\qquad U_3=K_3\exp(\Lambda(x))x^{\frac{\alpha}{2}(a-b)+1}W_2\left(\frac{W_3}{1-W_2-W_3}\right)^{\frac{1}{2}}\exp(-Q_\pm(W_1,W_2,W_3))$$

$$(3.25)\qquad U_4 = K_4 \exp(-\Lambda(x) - Q_\pm) x^{-\frac{\alpha}{2}(a-b)+1} \cdot \frac{-2(\beta - b + W_1)W_1(W_2 + W_3) + b(\beta - W_1)W_2}{2(\beta - b + W_1)W_1W_3 + (\beta + W_1)(2W_1 - b)W_2} \cdot \frac{W_3^{\frac{1}{2}}}{2(\beta - b + W_1)W_2(1 - W_2 - W_3)^{\frac{1}{2}}} + K_2 \frac{\beta - b + W_1}{\beta^2 - W_1^2} \exp(-\Lambda(x) + Q_\pm) x^{-\frac{\alpha}{2}(a-b)+1} \frac{-2(\beta - b + W_1)W_1(W_2 + W_3) + b(\beta - W_1)W_2}{2(\beta - b + W_1)W_2W_3^{\frac{1}{2}}(1 - W_2 - W_3)^{\frac{1}{2}}}$$

and

$$(1.30)\qquad V_1 = \left\{ W_1 - \frac{b(\beta - W_1)}{2(\beta - b + W_1)} W_2 \right\} (1 - W_2 - W_3)^{-1}$$

$$(1.31)\qquad V_2 = \frac{\beta^2 - W_1^2}{\beta - b + W_1} W_2 (1 - W_2 - W_3)^{-1}$$

$$(1.32)\qquad V_3 = \left\{ -W_1W_2 + \frac{b(\beta - W_1)}{2(\beta - b + W_1)} W_2 - W_1W_3 \right\} (1 - W_2 - W_3)^{-1},$$

where

$$(1.27)\qquad W_1 = C_1'$$

$$(1.28)\qquad W_2 = C_2' x^{\alpha(\beta + C_1')}$$

$$(1.29)\qquad W_3 = C_3' x^{2\alpha C_1'},$$

K_1, K_2, K_3, K_4, and C_1', C_2', C_3' being arbitrary constants.

Corollary The solution of (E) satisfying

$$(3.25)\qquad x^{-2}U_1U_2 = V_1, \qquad x^{-2}U_2U_3 = V_2, \quad x^{-2}U_3U_4 = V_3,$$

is obtained when the integration constants K_1, K_2, K_3, K_4, and C_1', C_2', C_3' satisfy

$$(3.26)\qquad K_1K_2 = C_1', \qquad K_2K_3 = \frac{\beta^2 - C_1'^2}{\beta - b + C_1'}, \quad K_4 = 0,$$

Proof It follows from (3.13) and (3.4) that

$$(3.27)\qquad x^{-2}U_1U_2 = K_1K_2 \frac{1}{1 - W_2 - W_3} - \frac{bK_2K_3}{2} \cdot \frac{W_2}{\beta + W_1} \cdot \frac{1}{1 - W_2 - W_3}$$

$$= \frac{1}{1-W_2-W_3}\left(W_1 - \frac{b}{2}\cdot\frac{\beta^2-W_1^{\,2}}{\beta-b+W_1}\cdot\frac{W_2}{\beta+W_1}\right)$$
$$= V_1$$

Similarly it follows from (3.4) and (3.7) that

$$(3.28)\qquad x^{-2}U_2U_3 = K_2K_3\frac{W_2}{1-W_2-W_3} = \frac{\beta^2-W_1^{\,2}}{\beta-b+W_1}\cdot\frac{W_2}{1-W_2-W_3} = V_2$$

And finally it follows from (3.7) and (3.25)

$$(3.29)\qquad x^{-2}U_3U_4 = \frac{K_2K_3(\beta-b+W_1)}{\beta^2-W_1^{\,2}}W_2\left(\frac{W_3}{1-W_2-W_3}\right)^{\frac{1}{2}} \cdot\frac{-2(\beta-b+W_1)W_1(W_2+W_3)+b(\beta-W_1)W_2}{2(\beta-b+W_1)W_2W_3^{\frac{1}{2}}(1-W_2-W_3)^{\frac{1}{2}}}$$
$$+K_3K_4\exp(-2Q_{\pm})\cdot W_2\left(\frac{W_3}{1-W_2-W_3}\right)^{\frac{1}{2}} \cdot\frac{-2(\beta-b+W_1)W_1(W_2+W_3)+b(\beta-W_1)W_2}{2(\beta-b+W_1)W_1W_3+(\beta+W_1)(2W_1-b)W_2}\cdot\frac{W_3^{\frac{1}{2}}}{2(\beta-b+W_1)W_2(1-W_2-W_3)^{\frac{1}{2}}}$$

$$(3.29)\qquad x^{-2}U_3U_4 = \frac{K_2K_3(\beta-b+W_1)}{\beta^2-W_1^{\,2}}\frac{1}{1-W_2-W_3}\cdot\frac{-2(\beta-b+W_1)W_1(W_2+W_3)+b(\beta-W_1)W_2}{2(\beta-b+W_1)}$$
$$= \frac{1}{1-W_2-W_3}\cdot\left\{-W_1(W_2+W_3)+\frac{b(\beta-W_1)W_2}{2(\beta-b+W_1)}\right\}$$
$$= V_3$$

References

[1] Noumi, M. and Yamada, Y., Higher order Painlevé equations of type $A_\ell^{(1)}$, Funkcial. Ekvac., **41** (1998) 483-503.

[2] Yoshida, S., A general solution of the simplified differential equations reduced from the 4th order Painlevé equation of type $A_\ell^{(1)}$, J. Sagami Women's Univ., **73B** (2009)1-10.

[3] Yoshida, S., A general solution of the simplified differential equations reduced from the 4th order Painlevé equation of type $A_\ell^{(1)}$,(II), ibid., **74B** (2010)43-52.